AP Calculus BC   Name:

Series Review

Note – Return and Review Team Quiz

Write the Maclaurin Series and interval of convergence for:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$ linear approx $(-\infty, \infty)$	$\frac{d}{dx}\sin x$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$ linear approx $(-\infty, \infty)$
$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n$ a_1 R $(-1, 1)$	integrate $\frac{1}{1-x}$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + \frac{(-1)^{n+1} x^n}{n}$ $(-1, 1]$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(-\infty, \infty)$	

Building Taylor Series and Taylor Polynomials

1. Write the 2nd degree Maclaurin polynomial for $h(x) = \sqrt[3]{x+2}$ and use it to approximate $\sqrt[3]{3}$

$h(0) = \sqrt[3]{2}$

$h'(0) = \frac{1}{3}(2)^{-2/3}$

$h''(0) = \frac{-2}{9}(2)^{-5/3}$

$P_2(x) = \sqrt[3]{2} + \frac{x}{3\sqrt[3]{4}} + \frac{-x^2}{9\sqrt[3]{32}}$

$\sqrt[3]{3} = \sqrt[3]{\underset{x}{1}+2} \approx \sqrt[3]{2} + \frac{1}{3\sqrt[3]{4}} - \frac{1}{9\sqrt[3]{32}} \approx 1.435$

2. Use the Lagrange error bound to find a range of possible values for $\sqrt[3]{3}$

$h'''(x) = \frac{10}{27}(x+2)^{-8/3}$

$\max h''' = \frac{10}{27}(2)^{-8/3} \approx 0.0583$

sub in
x = 0
x = 1

Possible values:

$1.435 \pm \frac{0.0583(1-0)^3}{3!}$

$= 1.435 \pm 0.00972$

3. Write the first four nonzero terms of the Taylor polynomial for $\sin x$ centered at $x = \frac{\pi}{6}$

$f(x)=\sin x$, $f(\frac{\pi}{6}) = \frac{1}{2}$

$f'(x) = \cos x$, $f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$, $f''(\frac{\pi}{6}) = -\frac{1}{2}$

$f'''(x) = -\cos x$, $f'''(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$

$P_4(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\frac{\pi}{6}) - \frac{1}{2 \cdot 2!}(x-\frac{\pi}{6})^2 - \frac{\sqrt{3}\,(x-\pi/6)^3}{2 \cdot 3!}$

4. Find the first four nonzero terms and the general term for the Maclaurin series for $f(x) = \frac{x^2}{1-x^4}$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$

$\frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + \ldots + x^{4n} + \ldots$

$\frac{x^2}{1-x^4} = x^2 + x^6 + x^{10} + x^{14} + \ldots + x^{4n+2} + \ldots$

5. Find the first four nonzero terms and the general term for the Maclaurin series for $f'(x)$

$f'(x) = 2x + 6x^5 + 10x^9 + 14x^{13} + \ldots + (4n+2)x^{4n+1} + \ldots$

6. Find the first four nonzero terms for the Maclaurin series for $F(x)$ if $F(0) = 1$

$F(x) = C + \frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \frac{x^{15}}{15} + \ldots + \frac{x^{4n+3}}{4n+3} + \ldots$

$F(0) = 1$, so... $C = 1$

$F(x) = 1 + \frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \ldots + \frac{x^{4n+3}}{4n+3}$ ← doesn't produce the linear term :/

7. Find the sum of the first 100 terms of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$n=-1: \quad 1 = -B \rightarrow B = -1$

$n = 0: \quad 1 = A$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$S_k = 1 - \frac{1}{k+1}$$

$$S_{100} = 1 - \frac{1}{101}$$

$$= \boxed{\frac{100}{101}}$$

8. Find the sum of $\sum_{n=1}^{\infty} \frac{4}{3^n}$, if it exists.

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n = \frac{4/3}{1-1/3} = \frac{4/3}{2/3} = \boxed{2}$$

9. Use the direct comparison test to determine convergence for $\sum_{k=1}^{\infty} \frac{7\cos^2 k}{k!}$ $= \sum a_k$

$\sum b_k = \sum_{k=1}^{\infty} \frac{7}{k^2}$ converges by p-series

$$\frac{7}{k^2} > \frac{7\cos^2 k}{k!}$$

$\therefore \sum_{k=1}^{\infty} \frac{7\cos^2 k}{k!}$ also converges by DCT

10. Use the limit comparison test to determine convergence for $\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+3)}$ $= \sum a_k$

compare to $\sum_{k=1}^{\infty} \frac{1}{k} = \sum b_k$, which diverges b/c it's the harmonic series

$$\lim_{k\to\infty}\left(\frac{1}{k} \cdot \frac{(k+1)(k+2)(k+3)}{k(k+3)}\right) = 1 \leftarrow \text{Finite \& positive}$$

$\therefore$ by LCT ~~by~~ $\sum a_k$ also converges

11. Use the ratio test to determine convergence for $\sum_{k=1}^{\infty} \frac{4^k}{(3k)!}$

$$\lim_{k\to\infty} \left| \frac{4^{k+1}}{(3k+3)!} \cdot \frac{(3k)!}{4^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{4}{(3k+3)(3k+2)(3k+1)} \right|$$

$$= |0| < 1$$

$\therefore$ the series converges by R.T.

12. Find the radius and interval of convergence for $\sum_{k=1}^{\infty} \frac{3^k}{k!} x^k$

$$\lim_{k\to\infty} \left| \frac{3^{k+1} x^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k \cdot x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{3x}{k+1} \right|$$

$$= |0| < 1 \text{ always}$$

radius $= \infty$

interval $= (-\infty, \infty)$

no need to "check endpoints"

13. Determine whether the series converges absolutely, converges conditionally, or diverges. $\sum_{k=1}^{\infty} \frac{(-1)^k (k+3)}{k(k+1)}$

A.S.T.

$\left\{ \frac{k+3}{k(k+1)} \right\}$ is positive & decreasing

$$\lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0$$

$\therefore$ converges by AST

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k (k+3)}{k(k+1)} \right| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)} = \sum a_k$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ is harmonic and diverges

$$\lim_{k\to\infty} \frac{k+3}{k(k+1)} \cdot \frac{k}{1} = 1 \leftarrow \text{finite \& pos}$$

$\therefore \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}$ also diverges by L.C.T.

$\therefore$ The series converges Conditionally

14. Use the alternating series error bound to show that the third order Maclaurin polynomial for $\ln(1+x)$ is within .02 units of $\ln(1+x)$ on [0, 0.5]

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

by A.S. error bound,

$$\text{Error} < \left| \frac{-x^4}{4} \right|_{x=0.5}$$

$$\text{Error} < 0.016 < 0.02$$