SOLUTIONS FOR REVIEW #9 - SERIES

1996 BC2
Solution

(a) $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{(n+1)!}$

$f'(0) = a_1 = \dfrac{1}{2}$

$f^{(17)}(0) = 17!a_{17} = 17!\left(\dfrac{1}{18!}\right) = \dfrac{1}{18}$

(b)

$$\lim_{n\to\infty} \frac{\left|\dfrac{x^{n+1}}{(n+2)!}\right|}{\left|\dfrac{x^n}{(n+1)!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$$

Converges for all x, by ratio test

(c) $g(x) = xf(x)$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

(d) $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$e^x - 1 = g(x) = xf(x)$

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

1997 BC2

Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.
   (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$f(1.5) \approx T_3(f,2)(1.5)$

$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$

$= -4.958\overline{3} = -4.958$

4 { 3: $T_3(f,2)(x)$
 <−1> each error
1: approximation of $f(1.5)$ }

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \ne -5$.

2 { 1: value of Lagrange Error Bound
1: explanation }

(c) $P(x) = T_4(g,0)(x)$

$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 { 2: $T_4(g,0)(x)$
 <−1> each incorrect, missing, or extra term
1: explanation }

Note:
<−1> max for improper use of $+\ldots$ or equality

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Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \ge 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1;\ f'(2) = \dfrac{2!}{3};\ f''(2) = \dfrac{3!}{3^2};\ f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 :
- 1 : coefficients $\dfrac{f^{(n)}(2)}{n!}$ in first four terms
- 1 : powers of $(x-2)$ in first four terms
- 1 : general term

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \dfrac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 :
- 1 : sets up ratio
- 1 : limit
- 1 : applies ratio test to conclude radius of convergence is 3

(c) $g(2) = 3;\ g'(2) = f(2);\ g''(2) = f'(2);\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 :
- 1 : first four terms
- 1 : general term

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

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Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

Solution	Points
(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	2 : { 1 : $f(2) = 7$ 1 : $f''(2) = -18$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	2 : { 1 : states $f'(2) = 0$ 1 : declares $f(2)$ as a relative maximum because $f''(2) < 0$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	3 : { 1 : $f(0) \approx T(0) = -5$ 1 : declares that it is not possible to determine 1 : reason
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0 - 2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	2 : { 1 : value of Lagrange error bound 1 : explanation

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim\limits_{x\to 0} \dfrac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

(a) $\lim\limits_{n\to\infty} \left| \dfrac{\dfrac{(n+2)x^{n+1}}{3^{n+2}}}{\dfrac{(n+1)x^n}{3^{n+1}}} \right| = \lim\limits_{n\to\infty} \left| \dfrac{(n+2)}{(n+1)}\dfrac{x}{3} \right| = \left| \dfrac{x}{3} \right| < 1$

At $x = -3$, the series is $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\sum\limits_{n=0}^{\infty} \dfrac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 : { 1 : sets up ratio test; 1 : computes limit; 1 : conclusion of ratio test; 1 : endpoint conclusion }

(b) $\lim\limits_{x\to 0} \dfrac{f(x) - \frac{1}{3}}{x} = \lim\limits_{x\to 0} \left(\dfrac{2}{3^2} + \dfrac{3}{3^3}x + \dfrac{4}{3^4}x^2 + \cdots \right) = \dfrac{2}{9}$

1 : answer

(c)
$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots \right) dx$$
$$= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \right) \Bigg|_{x=0}^{x=1}$$
$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$$

3 : { 1 : antidifferentiation of series; 1 : first three terms for definite integral series; 1 : general term }

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\int_0^1 f(x)\,dx = \dfrac{\frac{1}{3}}{1 - \frac{1}{3}} = \dfrac{1}{2}$.

1 : answer