BC Calculus

Review #4 – Differential Equations

1988 BC6 (no calculator)

Let f be a differentiable function for all $x \geq 0$ such that $f(0) = 5$ and $f(3) = -1$.

Suppose that for any number $b > 0$ the average value of $f(x)$ on the interval $0 \leq x \leq b$ is $\frac{f(0)+f(b)}{2}$.

$$\frac{1}{b}\int_0^b f(x)\,dx = \frac{f(0)+f(b)}{2}$$

a) Find $\int_0^3 f(x)\,dx$.

b) Prove that $f'(x) = \frac{f(x)-5}{x}$ for all $x > 0$.

c) Find $f(x)$.

a) $\int_0^3 f(x)dx$ ... use $b = 3$

$$\frac{1}{3}\int_0^3 f(x)\,dx = \frac{f(0)+f(3)}{2}$$

$$\int_0^3 f(x)\,dx = 3 \cdot \frac{5-1}{2} = 3 \cdot 2 = \boxed{6}$$

b)

$$\int_0^b f(x)\,dx = b \cdot \frac{f(0)+f(b)}{2}$$

$$F(b) - F(0) = \frac{b\left(f(0)+f(b)\right)}{2}$$

$$F(x) - F(0) = \frac{x\left(f(0)+f(x)\right)}{2}$$

$$F'(x) - 0 = \frac{\left(f(0)+f(x)\right) + x\left(f'(0)+f'(x)\right)}{2}$$

$$f(x) = \frac{(5+f(x)) + xf'(x)}{2}$$

$$2f(x) = 5 + f(x) + xf'(x)$$

$$f(x) - 5 = xf'(x)$$

$$\frac{f(x)-5}{x} = f'(x) \checkmark$$

c)

$$\frac{f(x)-5}{x} = f'(x)$$

$$\frac{y-5}{x} = \frac{dy}{dx}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y-5}$$

$$\ln|x| = \ln|y-5| + C$$

$$x = (y-5)\cdot C$$

$f(0)=5 \rightarrow 0 = (0)\cdot C$

$f(3)=-1 \rightarrow 3 = (-1-5)C$

$$3 = -6C$$

$$-\frac{1}{2} = C$$

$$x = -\frac{1}{2}(y-5)$$

$$-2x = y-5$$

$$\boxed{y = -2x+5}$$

1990 BC6 (no calculator)

Let f and g be continuous functions with the following properties.

i) $g(x) = A - f(x)$ where A is a constant.

ii) $\int_1^2 f(x)\,dx = \int_2^3 g(x).$

iii) $\int_2^3 f(x)\,dx = -3A.$

a) Find $\int_1^3 f(x)\,dx$ in terms of A.

b) Find the average value of $g(x)$ in terms of A, over the interval [1, 3].

c) Find the value of k if $\int_0^1 f(x+1)\,dx = kA.$

a) $\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$

$= \int_2^3 g(x) + -3A$

$= \int_2^3 (A - f(x))dx + -3A$

$= \int_2^3 A dx - \int_2^3 f(x)dx - 3A$

$= Ax\Big|_2^3 - (-3A) - 3A$

$= 3A - 2A$

$= \boxed{A}$

b) Avg value $= \frac{1}{3-1} \int_1^3 g(x)\,dx$

$= \frac{1}{2}\left[\cancel{\int_1^2 g(x)dx + \int_2^3 g(x)dx}\right]$

$= \frac{1}{2}\int_1^3 (A - f(x))dx$

$= \frac{1}{2}\int_1^3 A dx - \frac{1}{2}\int_1^3 f(x)\,dx$

$= \frac{1}{2}[3A - 1A] - \frac{1}{2}A$

$= \frac{1}{2}(2A) - \frac{1}{2}A$

$= \boxed{\frac{1}{2}A}$

c) $\int_0^1 f(x+1)dx = kA$

shift 1 unit

$\int_1^2 f(x)dx = kA$

$\int_2^3 g(x)\,dx = kA$

$\int_2^3 (A - f(x))\,dx = kA$

$\int_2^3 A\,dx - \int_2^3 f(x)\,dx = kA$

$3A - 2A - (-3A) = kA$

$4A = kA$

$\boxed{k = 4}$

1992 AB6 (no calculator)

At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius of the sphere is 2. (the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$).

$$\frac{dV}{dt} = \frac{k}{r} \qquad r(0) = 1 \quad r(15) = 2$$

a) Find the radius as a function of t.

b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = \frac{k}{r}$$

$$\int 4\pi r^3 dr = \int k\,dt$$

$$\pi r^4 = kt + C$$

$r(0) = 1 \rightarrow$ $\quad \pi = 0 + C$

$$\pi r^4 = kt + \pi$$

$r(15) = 2$ $\quad \pi \cdot 16 = 15k + \pi$

$$15\pi = 15k$$

$$\pi = k$$

$$\pi r^4 = \pi t + \pi$$

$$r^4 = t + 1$$

$$\boxed{r = \sqrt[4]{t+1}}$$

b) $V(t_0) = 27V(0)$

$$V(0) = \frac{4}{3}\pi\left(\sqrt[4]{t+1}\right)^3\Big|_{t=0}$$

$$V(0) = \frac{4}{3}\pi$$

$$27V(0) = 36\pi$$

$$V(t_0) = 36\pi$$

$$\frac{4}{3}\pi\left(\sqrt[4]{t_0+1}\right)^3 = 36\pi$$

$$\left(\sqrt[4]{t_0+1}\right)^3 = 27$$

$$\sqrt[4]{t_0+1} = 3$$

$$t_0 + 1 = 81$$

$$t_0 = 80 \qquad \boxed{\text{At } t = 80}$$

1998 AB4 (calculator)

Let f be a function with $f(1)=4$ such that for all points *(x, y)* on the graph of *f*, the slope is given by $\frac{3x^2+1}{2y}$. $= \frac{dy}{dx}$

a) Find the slope of the graph of f at the point where $x=1$.

b) Write an equation for the line tangent to the graph of f at $x=1$ and use it to approximate *f(1.2)*.

c) Find *f(x)* by solving the separable differential equation $\frac{3x^2+1}{2y}$ with the initial condition that $f(1)=4$.

d) Use your solution from part c) to find *f(1.2)*.

a) $\frac{dy}{dx} = \frac{3(1)^2+1}{2(4)} = \frac{4}{8} = \boxed{\frac{1}{2}}$

b) slope $= \frac{1}{2}$ point $= (1,4)$

$y - 4 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}(x-1) + 4$

$y(1.2) = \frac{1}{2}(1.2-1)+4$

$= \frac{1}{2}\left(\frac{1}{5}\right)+4$

$= \boxed{4.1}$

c) $\frac{dy}{dx} = \frac{3x^2+1}{2y}$

$\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$, $f(1)=4$

$4^2 = 1+1+C$

$16 = 2 + C$

$14 = C$

$y^2 = x^3 + x + 14$

$\boxed{y = \sqrt{x^3+x+14}}$

d) $y(1.2) = \sqrt{1.2^3 + 1.2 + 14}$

$\approx \boxed{4.114}$