BC Calculus

Review #2 - Derivatives

Differentiate the following

1) $f(x) = x^{14} + 10x^2 + 7x - 4$

$f'(x) = 14x^{13} + 20x + 7$

2) $y = x^{-2} - 3x^{-5} + 3x^{-8}$

$y' = -2x^{-3} + 15x^{-6} - 24x^{-9}$

3) $g(x) = \dfrac{x^3 - 3x + 5}{x^2}$

$g(x) = x - 3x^{-1} + 5x^{-2}$

$g'(x) = 1 + 3x^{-2} - 10x^{-3}$

4) $f(x) = \cos\left(\dfrac{x-1}{x+1}\right)$

$f'(x) = -\sin\left(\dfrac{x-1}{x+1}\right)\left[\dfrac{1(x+1)-(x-1)}{(x+1)^2}\right]$

$= -\sin\left(\dfrac{x-1}{x+1}\right)\left(\dfrac{2}{(x+1)^2}\right)$

5) $k(x) = \dfrac{(3x-2)^6}{(2x+1)^7}$

$k'(x) = \dfrac{6(3x-2)^5(2x+1)^7(3) - (3x-2)^6 \cdot 7 \cdot 2(2x+1)^6}{(2x+1)^{14}}$

6) $f(x) = \ln\left(x^{\frac{3}{2}} + 2x^{\frac{3}{5}} - 4x^{\frac{4}{7}}\right)$

$f'(x) = \dfrac{\frac{3}{2}x^{1/2} + \frac{6}{5}x^{-2/5} - \frac{16}{7}x^{-3/7}}{x^{3/2} + 2x^{3/5} - 4x^{4/7}}$

7) $F(x) = \arctan(4x+3)$

$f(x) = \dfrac{1}{(4x+3)^2+1} \cdot 4$

8) $f(x) = \sin^3(3x^2)e^{-4x}$

$f'(x) = 3\sin^2(3x^2) \cdot \cos(3x^2) \cdot 6x e^{-4x}$

$+ \sin^3(3x^2)(-4e^{-4x})$

9) $f(x) = \cot(3x)\sec^4(2x)$

$f'(x) = \dfrac{\cot(3x)}{\cos^4(2x)}$

$= \dfrac{-\csc^2(3x) \cdot 3\cos^4(2x) - \cot(3x) \cdot 4\cos^3(2x)(-\sin(2x)) \cdot 2}{\cos^8(2x)}$

10) $y = (\sin x)^{\tan x}$

$\ln y = \tan x \ln(\sin x)$

$\dfrac{y'}{y} = \sec^2 x \ln(\sin x) + \dfrac{\tan x \cos x}{\sin x}$

$y' = (\sin x)^{\tan x} \cdot$ (that stuff above ↑)

For #15-16, Find dy/dx

11) $x^2 - \ln(2xy) + 3y^2 = 2$

$$2x - \frac{1}{2xy}\cdot[2y + 2xy'] + 6y\cdot y' = 0$$

$$\frac{2xy'}{2xy} + 6y\cdot y' = -2x + \frac{2y}{2xy}$$

$$y'\left(\frac{1}{y} + 6y\right) = -2x + \frac{1}{x}$$

$$y' = \frac{-2x + \frac{1}{x}}{\frac{1}{y} + 6y}$$

So fun thank you! :)

12) $y^2 + (3\cos x)y^2 + 3x^3 - 5 = 0$

$$2y\cdot y' + -3\sin x\, y^2 + (3\cos x)(2y)y' + 9x^2 = 0$$

$$y'(2y + 6y\cos x) = 3\sin x\cdot y^2 - 9x^2$$

$$y' = \frac{3\sin(x)\, y^2 - 9x^2}{2y + 6y\cos x}$$

13) Write an equation for the lines tangent to and normal to the graph of $4x^2 + 9y^2 = 36$ when ~~x=36~~.

$x=0$ and $y=2$

$$4x^2 + 9y^2 = 36$$

$$8x + 18y\cdot y' = 0$$

$$y' = \frac{-8x}{18y}$$

$$x = 0 \rightarrow 4(\cancel{2x})\,0 + 9y^2 = 36$$

$$9y^2 = 36$$

$$y^2 = 4$$

$$y = 2 \text{ or } -2$$

$$y' = \frac{-8x}{18y}$$

$$y' = \frac{-8(0)}{18(2)} = 0 \leftarrow \text{horizontal line}$$

@ $\boxed{y=2}$

Normal: $\boxed{x=0}$

14) Given the parametric function $x = 3t - 5$ and $y = t^2 + 2t - 4$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\boxed{\frac{dy}{dx} = \frac{2t+2}{3}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{2}{3}t + \frac{2}{3}\right)}{3} = \frac{\frac{2}{3}}{3} = \frac{2}{9}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{2}{9}}$$

1992 BC3 (calculator active)

At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$.

a) Find the slope of the path of the particle when $t = \frac{\pi}{2}$.

b) Find the speed of the particle when $t = 1$.

c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 1$.

a) Slope $= \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \left.\frac{e^t\cos t + e^t(-\sin t)}{e^t\sin t + e^t\cos t}\right|_{t=\frac{\pi}{2}}$

$= \frac{e^{\pi/2}(0) - e^{\pi/2}(1)}{e^{\pi/2}(1) + e^{\pi/2}(0)} = \frac{-e^{\pi/2}}{e^{\pi/2}} = \boxed{-1}$

b) speed $= \left.\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right|_{t=1}$

$= \sqrt{(e\cos 1 - e\sin 1)^2 + (e\sin 1 + e\cos 1)^2}$

calculator

$= 3.844$

c) Distance traveled $= \int_0^1 \sqrt{(e^t\sin t + e^t\cos t)^2 + (e^t\cos t - e^t\sin t)^2}\, dt$

$= 5.977$

1996 AB4 BC4 (calculator allowed)

This problem deals with functions by $f(x) = x + b\sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

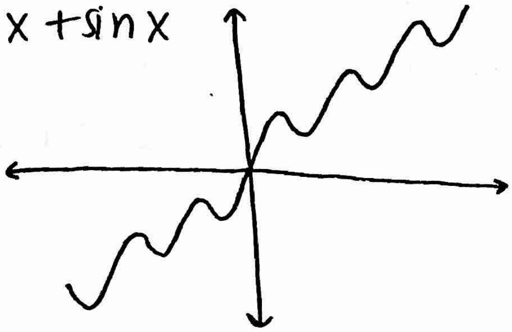
a) Sketch the graph of the two functions, $y = x + \sin x$ and $y = x + 3\sin x$.

b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b\sin x$.

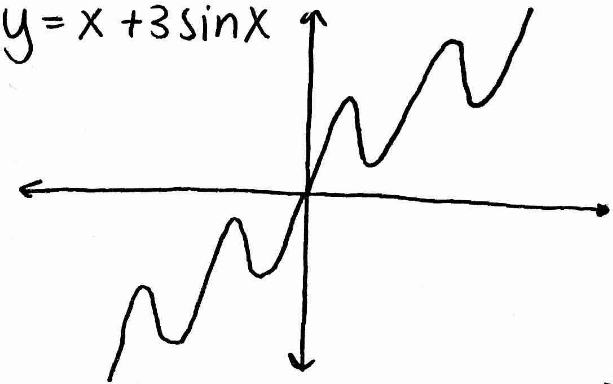
c) Are the points of tangency described in part b) relative maximum points of f? Why?

d) For all values of $b > 0$, show that all inflection points of the graph of f lie on the line $y = x$.

a) $y = x + \sin x$



$y = x + 3\sin x$



b) $y = x + b$ tangent to $f(x) = x + b\sin x$

$\uparrow$

slope $= 1$

slope $= f'(x) = 1 + b\cos x = 1$

$b\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2} + \pi k$

on $[-2\pi, 2\pi]$ $\boxed{x = \left\{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}}$

c) $x = \left\{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

$f'(x) = 0$ when $x = \left\{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$. Only those where f' changes from positive to negative are rel. maxima of f.

f': + + + + + on the number line with marks at $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$

$\therefore$, None of them are rel. maxima

d) $f''(x) = -b\sin x = 0$

$x = \pi k$

$f(\pi k) = \pi k + b\sin(\pi k)$

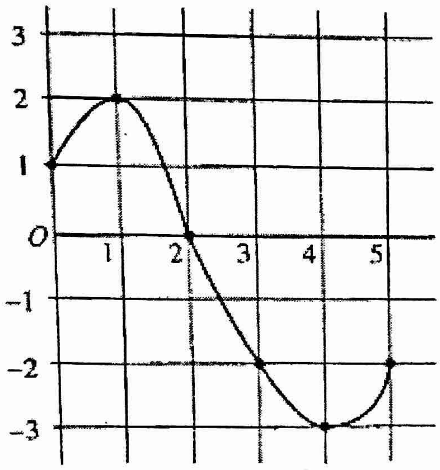
$f(\pi k) = \pi k$

$f(x) = x$

$y = x$ ✓

1995 BC6

Let *f* be a function whose domain is the closed interval [0, 5]. The graph of *f* is shown below.



Graph of f

a) $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$ domain of f $[0,5]$

$0 \le \frac{x}{2}+3 \le 5$

$-3 \le \frac{x}{2} \le 2$

$\boxed{-6 \le x \le 4}$

Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$.

a) Find the domain of *h*.

b) Find $h'(2)$.

c) At what *x* is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

b) $h'(x) = f\left(\frac{x}{2}+3\right) \cdot \frac{1}{2}$

$h'(2) = f(4) \cdot \frac{1}{2}$

$= -3 \cdot \frac{1}{2}$

$= \boxed{-\frac{3}{2}}$

c) $h'(x) = f\left(\frac{x}{2}+3\right) \cdot \frac{1}{2} = 0$

$f\left(\frac{x}{2}+3\right) = 0$, since $f(2)=0$,

$\frac{x}{2}+3 = 2$

$\frac{x}{2} = -1$

$x = -2$

$h(-6) = \int_0^0 f(t)dt = 0$

$h(4) = \int_0^5 f(t)dt < 0$

since the area below the x-axis is greater than the area above

$h(-2) = \int_0^2 f(t)dt > 0$

$\therefore$ The minimum value is at $\boxed{x=4}$

1996 AB4 BC4 (calculator allowed)

This problem deals with functions by $f(x) = x + b\sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

a) Sketch the graph of the two functions, $y = x + \sin x$ and $y = x + 3\sin x$.

b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b\sin x$.

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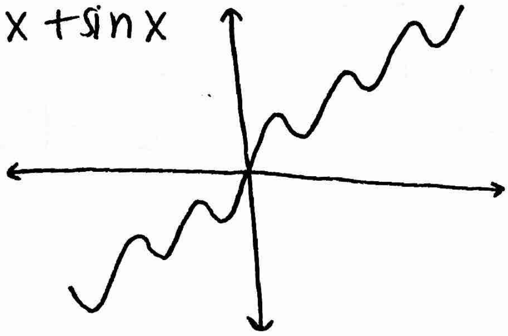

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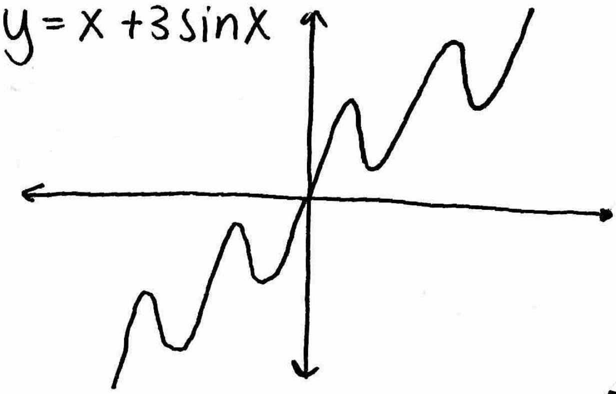

b) $y = x + b$ tangent to $f(x) = x + b\sin x$

Slope $= 1$

$$\text{slope} = f'(x) = 1 + b\cos x = 1$$
$$b\cos x = 0$$
$$\cos x = 0$$
$$x = \frac{\pi}{2} + \pi k$$

on $[-2\pi, 2\pi]$ $\boxed{x = \left\{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}}$

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f': + + + + + at $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$

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$x = \pi k$

$f(\pi k) = \pi k + b\sin(\pi k)$
$f(\pi k) = \pi k$
$f(x) = x$
$y = x$ ✓

1995 BC6

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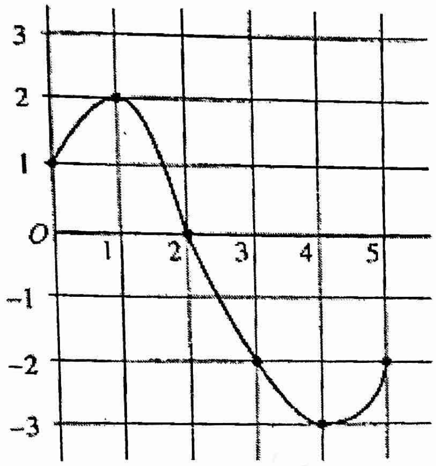

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Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$.

a) Find the domain of h.

b) Find $h'(2)$.

c) At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

b) $h'(x) = f\left(\frac{x}{2}+3\right) \cdot \frac{1}{2}$

$h'(2) = f(4) \cdot \frac{1}{2}$

$= -3 \cdot \frac{1}{2}$

$= \boxed{\frac{-3}{2}}$

c) $h'(x) = f\left(\frac{x}{2}+3\right) \cdot \frac{1}{2} = 0$

$f\left(\frac{x}{2}+3\right) = 0$, since $f(2) = 0$,

$\frac{x}{2}+3 = 2$

$\frac{x}{2} = -1$

$x = -2$

$h(-6) = \int_0^0 f(t)dt = 0$

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since the area below the x-axis is greater than the area above

$h(-2) = \int_0^2 f(t)dt > 0$

$\therefore$ The minimum value is at $\boxed{x = 4}$

1991 BC4 (no calculator)

A particle moves along the x-axis so that at time t its position is given by $x(t) = \sin\left(\pi t^2\right)$ for $-1 \le t \le 1$.

a) Find the velocity at time t.

b) Find the acceleration at time t.

c) For what values of t does the particle change direction?

d) Find all values of t for which the particle is moving left.

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a) $x(t) = \sin(\pi t^2)$

$v(t) = x'(t) = 2\pi t \cos(\pi t^2)$

b) $a(t) = v'(t) = 2\pi \cos(\pi t^2) + 2\pi t \cdot 2\pi t(-\sin(\pi t^2))$

c) "change direction" → velocity changes sign

$v(t) = 2\pi t \cos(\pi t^2) = 0$

$2\pi t = 0$ $\quad \cos(\pi t^2) = 0$

$t = 0$ $\quad \pi t^2 = \frac{\pi}{2} + \pi k$

$t^2 = \frac{1}{2} + k$

$t = \pm\sqrt{\frac{1}{2} + k}$

on $[-1,1]$, $t = \pm\sqrt{\frac{1}{2}}$

$v(t)$

t: $+$ | $-\sqrt{\frac{1}{2}}$ | $-$ | 0 | $+$ | $\sqrt{\frac{1}{2}}$ | $-$

since v changes sign at $t = \{0, \pm\sqrt{\frac{1}{2}}\}$, the particle changes direction then.

d) "Moving left" → velocity is negative

v is negative on $\left(-\sqrt{\frac{1}{2}}, 0\right)$ and $\left(\sqrt{\frac{1}{2}}, 1\right]$

1994 BC4 (no calculator)

A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = t\ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

a) Write an expression for the acceleration of the particle.

b) For what values of t is the particle moving to the right?

c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

d) Write an expression for the position, $x(t)$, of the particle.

a) $v(t) = t\ln t - t$

$a(t) = \ln t + 1 - 1 = \ln t$

$\boxed{a(t) = \ln t}$

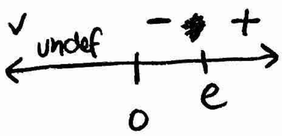
b) Moving to the right: $v(t) > 0$

$t\ln t - t > 0$

$t(\ln t - 1) = 0$

$t = 0 \quad \ln t = 1$

$t = e$

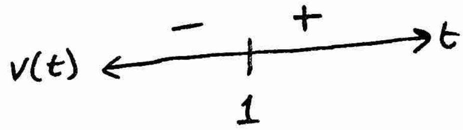


Moving to the right on $\boxed{t > e}$

c) Minimum velocity: $v'(t) = a(t) = \ln t$

$\ln t = 0$

$t = 1$



Because v' changes from negative to positive at $t = 1$, v has a relative minimum there.

$v(1) = 1\ln 1 - 1 = 0 - 1 = -1$

at $t = 1$, the velocity is $\boxed{-1}$

d) $x(t) = \int v(t)\,dt$

$= \int (t\ln t - t)\,dt$

$= \int t\ln t\,dt - \int t\,dt$

$u = \ln t \quad dv = t\,dt$

$du = \frac{1}{t}dt \quad v = \frac{1}{2}t^2$

$= \frac{1}{2}t^2\ln t - \int \frac{1}{2}t^2 \cdot \frac{1}{t}\,dt$

$= \frac{1}{2}t^2\ln t - \frac{1}{2}\int t\,dt$

$x(t) = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2 + C, \quad x(1) = 6$

$6 = \frac{1}{2}(1)(0) - \frac{1}{4} + C$

$\frac{25}{4} = C$

$\boxed{x(t) = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2 + \frac{25}{4}}$