AP Calculus BC

Name:

Tools for Testing Convergence of a Power Series

Since a power series, $\sum_{n=k}^{\infty} a_n$, is a good approximation only on its interval of convergence, we need to develop some tools to find the interval. We always check the interior of the interval first, then, we check the endpoints for convergence. There are several useful tests. Each test has conditions that must be checked before the test can be applied.

*n*th-term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ or fails to exist, then $\sum_{n=k}^{\infty} a_n$ diverges.

Example 1. Determine if $\sum_{n=1}^{\infty} \frac{n-1}{n+1}$ could be convergent.

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1 \neq 0$$

The series diverges by the n^{th} term test.

Geometric Series

A geometric series in the form $\sum_{n=k}^{\infty} (r)^n$ converges only if $-1 < r < 1$.

The sum of a convergent series is $S = \frac{a_k}{1-r}$.

Example 2. Verify that $\sum_{n=1}^{\infty} \left(\frac{-3}{5}\right)^n$ converges. Find the sum of the series as $n \to \infty$.

This is geometric with $|r| < 1$

$\therefore$ the series converges to $\dfrac{\frac{-3}{5}}{1+\frac{3}{5}} = \boxed{\frac{-3}{8}}$

AP Calculus BC

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p-Series Test

p-series are an extension of the Integral Test discussed in the chapter on improper integrals.

used to justify the test

If a_n is in the form $\frac{1}{n^p}$,

Then

a) $\sum_{n=k}^{\infty} \frac{1}{n^p}$ will diverge if $p \le 1$.

b) $\sum_{n=k}^{\infty} \frac{1}{n^p}$ will converge if $p > 1$.

Example 3. Does $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converge?

$= 3\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\quad \frac{3}{2} > 1$ This series converges by the p-series test.

Direct Comparison Test ← you can usually go directly to the limit comparison test

This test is used when a known series is bigger than the given series.

Part 1

If a_n has no negative terms,

and a ceiling function $\sum_{n=k}^{\infty} b_n$ converges, then $\sum_{n=k}^{\infty} a_n$ must also converge.

Part 2

If a_n has no negative terms,

and a floor function $\sum_{n=k}^{\infty} b_n$ diverges, then $\sum_{n=k}^{\infty} a_n$ must also diverge.

Example 4. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges.

$\frac{1}{n^2+3} < \frac{1}{n^2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p series test

$\therefore \sum_{n=1}^{\infty}$ also converges by the D.C.T.

Limit Comparision Test

This is one of the most useful tests for determining convergence.

Suppose $a_n > 0$ and $b_n > 0$ for all $n \geq N$ where N is a positive integer.

i.e. @ SOME point, the terms become all positive. and >1

a. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ behave the same.

b. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ must also converge.

c. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ must also diverge.

Example 5. Determine if $\sum_{n=1}^{\infty} \frac{3n+2}{(n+1)^2}$ converges.

$a_n =$

$b_n = \sum_{n=1}^{\infty} \frac{3n}{n^2} = \sum_{n=1}^{\infty} \frac{3}{n}$

$$\lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{3n+2}{(n+1)^2}} = \lim_{n\to\infty} \frac{3(n+1)^2}{n(3n+2)} = 1$$

since $0<1<\infty$, b_n and a_n behave the same.

By the p-series test, $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges, $\therefore \sum_{n=1}^{\infty} \frac{3n+2}{(n+1)^2}$ also diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{3}{n}$

Example 6. Determine if $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

$b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges because Geometric series with $|R|<1$.

$$\lim \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = 1.$$

they behave the same

By the limit comparison test with $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ also converges.

Example 7. Determine if $\sum_{n=1}^{\infty} \frac{2n+1}{n^3 - 2n}$ converges.

let $a_n = \frac{2n+1}{n^3-2n}$

$b_n = \frac{2}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)(n^2)}{(n^3-2n)(2)} = 1$$

By the p-series test, $\sum b_n$ converges.

since this limit is a positive constant, $\sum_{n=1}^{\infty} a_n$ and $\sum b_n$ behave the same.

$\therefore$ By the limit comparison test with $\sum_{n=1}^{\infty} \frac{2}{n^2}$, $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-2n}$ also converges.

SUPER TEST!

Ratio Test

This test is used most often on the AP Exam to determine convergence of a power series.

Let $\sum_{n=k}^{\infty} a_n$ be a series with positive terms and that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$ ← common ratio... ish!

↑ i.e. it is approaching a geometric series

Then,

a) If $L<1,$ the series converges.

b) if $L>1,$ the series diverges.

c) if $L=1,$ the test fails. Use some other test.

Example 8. Use the Ratio test to determine if the series $\sum_{n=0}^{\infty}\frac{2^n}{3^n+1}$ converges.

$$a_n = \frac{2^n}{3^n+1}$$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{3^{n+1}+1}}{\frac{2^n}{3^n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(3^n+1)}{2^n(3^{n+1}+1)}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{3^n}{3^{n+1}}\right| = \left|\frac{2}{3}\right| < 1$$

$$a_{n+1} = \frac{2^{n+1}}{3^{n+1}+1}$$

since $\left|\frac{2}{3}\right|<1$, $\sum_{n=0}^{\infty}\frac{2^n}{3^n+1}$ converges by the ratio test.

Example 9. Use the ratio test to determine the radius of convergence for the series $\sum_{n=0}^{\infty}\frac{nx^n}{10^n}.$ ← see "x"... go to Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{10^{n+1}}}{\frac{nx^n}{10^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{10^{n+1}}\cdot\frac{10^n}{nx^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)\cdot\left(\frac{x^{n+1}}{x^n}\right)\left(\frac{10^n}{10^{n+1}}\right)\right|$$

1 point

$$= \left|1\cdot x\cdot\frac{1}{10}\right|$$

$$= \left|\frac{x}{10}\right|$$ ← 1 point

The series converges if $\left|\frac{x}{10}\right| < 1$

$$-1 < \frac{x}{10} < 1$$

$$-10 < x < 10$$ ← 1 point

The radius of convergence is 10

Example 10. Find the interval of convergence for $\sum_{n=0}^{\infty}(x+5)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+5)^n\cdot(x+5)}{(x+5)^n}\right| = |x+5|$$

converges when $|x+5|<1$

$-1<x+5<1$

$-6<x<-4$

radius of convergence is 1

Interval of convergence is $(-6,-4)$

check endpoints $\sum_{n=0}^{\infty}(-4+5)^n = \sum_{n=0}^{\infty}1^n$ diverges by n^{th} term test

$\sum_{n=0}^{\infty}(-6+5)^n = \sum_{n=0}^{\infty}(-1)^n$ oscillates/diverges.

Alternating Series Test

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ converges if all three of the following conditions are met.

1. Each term of the series is positive; $\{u_1, u_2, u_3, u_4, \ldots\}$ are all positive
2. Eventually, the terms decrease in value; (absolute)
3. $\lim_{n\to\infty}u_n = 0$. decrease to 0.

$-4\le x<4$

This test is useful in checking convergence at an endpoint of an interval after the Ratio Test determines the interior of an interval.

Example 11. Find the interval of convergence for $\sum_{n=4}^{\infty}\frac{nx^n}{4^n(n^2+1)}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{4^{n+1}\left[(n+1)^2+1\right]}\cdot\frac{4^n(n^2+1)}{nx^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)\left(\frac{x^{n+1}}{x^n}\right)\left(\frac{4^n}{4^{n+1}}\right)\left(\frac{n^2+1}{n^2+2n+2}\right)\right|$$

$$= \left|1\cdot x\cdot\left(\tfrac{1}{4}\right)\cdot 1\right|$$

$$= \left|\frac{x}{4}\right|$$

Converges when $-1<\frac{x}{4}<1$

$-4<x<4$ ..... now check endpoints!

let $x=-4$

$\sum_{n=4}^{\infty}\frac{(-4)^n(n)}{4^n(n^2+1)} = \sum_{n=4}^{\infty}(-1)^n\left(\frac{n}{n^2+1}\right)$ alternating!

1. $\frac{n}{n^2+1}$ are positive for all $n\ge 4$
3. $\lim_{n\to\infty}\frac{n}{n^2+1}=0$ $\frac{n}{n^2+1}$ is decreasing

$\Rightarrow$ converges by alt. ser. test

let $x=4$

$\sum_{n=4}^{\infty}\frac{n}{n^2+1}$ use limit comparison test

... must complete

$\boxed{-4\le x<4}$

Ratio Test: Example 12. Find the interval of convergence for $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{2n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{2n+2}}{2n+2}\cdot\frac{2n}{(-1)^{n+1}x^{2n}}\right| = \left|\frac{(-1)x^2}{1}\right| = |-x^2| = x^2$$

The series converges by the ratio test when $-1 < x^2 < 1$

$\Rightarrow \quad 0 \leq x^2 < 1$ **

$-1 < x < 1$

Check Endpoints:

$x = -1$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^{2n}}{2n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}$$

converges by A.S.T.

$x = 1$

$\vdots$

converges by A.S.T.

$$\boxed{-1 \leq x \leq 1}$$

Example 13. Find the interval of convergence for $\sum_{n=4}^{\infty}\frac{(x-2)^n}{3n}$.

Ratio Test (does it approach a converging geometric?)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{3n+3}\cdot\frac{3n}{(x-2)^n}\right| = \left|\frac{x-2}{1}\right|$$

The series converges by the ratio test when $|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

Check Endpoints:

$x = 1$:

$$\sum_{n=4}^{\infty}\frac{(1-2)^n}{3n} = \sum_{n=4}^{\infty}\frac{(-1)^n}{3n}$$

converges by AST

$x = 3$:

$$\sum_{n=4}^{\infty}\frac{(3-2)^n}{3n} = \sum_{n=4}^{\infty}\frac{1}{3n}$$

Diverges by p-series test

$$\boxed{1 \leq x < 3}$$

Example 14. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n}$.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\sqrt{n+1}\cdot 3^{n+1}} \cdot \frac{n\sqrt{n}\,3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$= \left| x \cdot 1 \cdot 1 \cdot \tfrac{1}{3} \right|$$

$$= \left| \frac{x}{3} \right| < 1$$

Converges when

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

Check Endpoints

$x = -3$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}3^n}$

Converges by AST

$x = 3$ $\sum_{n=1}^{\infty} \frac{3^n}{n\sqrt{n}3^n}$

Converges by p-series test

$$\boxed{-3 \le x \le 3}$$

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Example 15. Determine if $\sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ converges.

$$\sum_{n=1}^{\infty} \left| \frac{(\sin n)^n}{n^2} \right|$$

$$\frac{|\sin n|^n}{n^2} < \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-series

$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(\sin n)^n}{n^2} \right|$ also conv. by D.C.T.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ absolutely converges.

Example 16. Does the series $\frac{1}{3} - \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$? converge?

$$\left|\frac{1}{3}\right| + \left|-\frac{1}{9}\right| + \left|\frac{-1}{27}\right| + \left|\frac{1}{81}\right| + \left|\frac{1}{243}\right| + \ldots$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$$

$= \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$ which is geometric w/ $|r| < 1$ and converges

$\therefore \frac{1}{3} - \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \ldots$ converges absolutely

Example 17. Given the series $\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$. Determine if it is conditionally convergent, absolutely convergent, or neither.

let $\sum a_k = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots$

$$= \sum_{k=3}^{\infty} (-1)^{k+1} \frac{1}{k}$$

$\sum |a_k| = \sum_{k=3}^{\infty} \frac{1}{k}$ which is harmonic & diverges

<u>BUT</u> $\sum a_k$ is alternating, with $\left\{\frac{1}{k}\right\}$ positive, $\lim_{k \to \infty} \frac{1}{k} = 0$ $\Rightarrow$ converges conditionally by A.S.T.