Example 1 – calculator allowed

Consider the parametric function $\begin{cases} x = t^2 - 4 \\ y = 3\sin t \end{cases}$ $for\ 0 \le t \le \pi.$

b) Find the highest point on the curve. (x, y)

largest (max) y-value

maximize $y = 3\sin t$ on $[0, \pi]$ ← Closed. E.V.T.

Crit. values: $y' = 3\cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2} + \pi k$

on $[0, \pi]$, $\frac{\pi}{2}$

Test:

$y(0) = 3\sin 0 = 0$

$y(\pi) = 3\sin \pi = 0$

$y(\frac{\pi}{2}) = 3\sin(\frac{\pi}{2}) = \underline{\underline{3}}$

$x(\frac{\pi}{2}) = (\frac{\pi}{2})^2 - 4$

$= \frac{\pi^2}{4} - 4$

$\boxed{(\frac{\pi^2}{4} - 4, 3)}$

c) Find the length of the curve on the interval.

$\int_0^\pi \sqrt{(2t)^2 + (3\cos t)^2}\, dt \approx \boxed{12.418 \text{ units}}$

Example 2 – NO calculator

Given the parametric equation $\begin{cases} x = 2t^2 + 3 \\ y = t^4 \end{cases}$, write the equation for the line tangent to the path of the function when $t = -1$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2$ ← slope equation

$\left.\frac{dy}{dx}\right|_{t=-1} = (-1)^2 = 1$ ← slope of tangent line

$x(-1) = 2 + 3 = 5$

$y(-1) = 1$

$(5, 1)$ ← point on curve @ $t = -1$

$\boxed{y - 1 = 1(x - 5)}$ ← point-slope form

Example 3 – NO calculator

Given the parametric function $\begin{cases} x = \dfrac{1}{t} \\ y = -2 + \ln t \end{cases}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of *t*.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/t}{-1/t^2} = -t \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-1}{-1/t^2} = t^2$$

$$\boxed{\frac{dy}{dx} = -t} \qquad \boxed{\frac{d^2y}{dx^2} = t^2}$$

998 BC6 - calculator allowed

A particle moves along the curve defined by the equation $y = x^3 - 3x$. The *x*-coordinate of the particle, *x(t)*, satisfies the equation $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{2t+1}}$ for $t \geq 0$ with initial condition that $x(0) = -4$.

a) Find $x(t)$ in terms of *t*.

$$\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$$

$$\int dx = \int \frac{dt}{\sqrt{2t+1}} \qquad u = 2t+1, \quad \tfrac{1}{2}du = dt$$

$$x = \frac{1}{2}\int u^{-1/2}\,du$$

$$x = \frac{1}{2} \cdot 2u^{1/2} + C$$

$$x = \sqrt{2t+1} + C, \quad x(0) = -4$$

$$-4 = \sqrt{1} + C$$

$$-5 = C$$

$$\boxed{x(t) = \sqrt{2t+1} - 5}$$

b) Find $\frac{dy}{dt}$ in terms of t.

$y = x^3 - 3x$

$y = (\sqrt{2t+1} - 5)^3 - 3(\sqrt{2t+1} - 5)$

$\frac{dy}{dt} = 3(\sqrt{2t+1} - 5)^2 \left(\frac{1}{\sqrt{2t+1}}\right) - \frac{3}{\sqrt{2t+1}}$

$= \boxed{\frac{3(\sqrt{2t+1} - 5)^2 - 3}{\sqrt{2t+1}}}$

c) Find the location and speed of the particle at time $t = 4$.

Location:

$x(4) = \sqrt{9} - 5 = -2$

$y(x) = x^3 - 3x$

$y(-2) = (-2)^3 - 3(-2)$

$= -8 + 6$

$= -2$

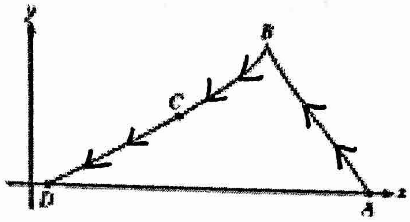
$\boxed{(-2, -2)}$

speed: $\left.\frac{dy}{dt}\right|_{t=4} = \frac{3(3-5)^2}{3} - \frac{3}{3} = 4 - 1 = 3$

$\left.\frac{dx}{dt}\right|_{t=4} = \frac{1}{3}$

speed $= \sqrt{\frac{1}{9} + 9} = \frac{\sqrt{82}}{3} = \boxed{3.018}$

2003 BC2 (calculator allowed)



$\leftarrow 0 \le t \le 9$

A particle starts at point *A* on the positive *x*-axis at time $t \ge 0$ and travels along the curve from *A* to *B* to *C to* D, as shown above. The coordinates of the particle's position $(x(t), y(t))$ are differentiable functions of *t*, where $x'(t) = \frac{dx}{dt} = -9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \frac{dy}{dt}$ is not explicitly given. At time *t* = 9, the particle reaches its final position *D* on the positive *x*-axis.

a) At point *C*, is $\frac{dy}{dt}$ positive? At point *C*, is $\frac{dx}{dt}$ positive? Give a reason for each answer.

$\frac{dy}{dt}$ and $\frac{dx}{dt}$ are both negative, because the particle is traveling down (y decreasing) and to the left (x decreasing) @ point C.

b) The slope of the curve is undefined at point *B*. At what time is the particle at point *B*?

$$\text{Slope} = \frac{dy}{dx} = \frac{dy/dt}{-9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right)}$$ $\leftarrow$ undefined when denominator = 0

$$-9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) = 0$$

$\cos\left(\frac{\pi t}{6}\right) = 0$

$\frac{\pi t}{6} = \frac{\pi}{2} + \pi k$

$t = 3 + 6k$

on $[0, 9]$,

$t = 3$

$\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) = 0$

$\frac{\pi\sqrt{t+1}}{2} = \pi k$

$\sqrt{t+1} = 2k$

$t + 1 = 4k^2$

$t = -1 + 4k^2$

$t = \{3, 15, \ldots\}$

$t = 3$ on $[0, 9]$

$\boxed{\text{at time } t = 3\text{, the particle is at point B}}$

c) The line tangent to the curve at the point $(x(8), y(8))$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

slope @ $t=8$ $= \frac{5}{9} = \frac{y'(8)}{x'(8)}$

$\left.\frac{dx}{dt}\right|_{t=8} = -9\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{3\pi}{2}\right)$

$= -9\left(-\frac{1}{2}\right)(-1)$

$= -\frac{9}{2}$

$\frac{5}{9} = \frac{y'(8)}{-9/2}$

$y'(8) = \frac{-5}{2}$

$\boxed{\text{velocity} = \left\langle \frac{-9}{2}, \frac{-5}{2} \right\rangle}$

Speed $= \sqrt{\left(\frac{-9}{2}\right)^2 + \left(\frac{-5}{2}\right)^2}$

$\boxed{\approx 5.148}$

d) How far apart are points *A* and *D*, the initial and final positions, respectively, of the particle?

$y(0) = y(9) = 0$

Displacement btwn points A & D (time 0 & 9)

$= x(9) - x(0)$

$= \int_0^9 x'(t)\,dt$ ← use calculator

≈ -39.255

The positions are 39.255 units apart.