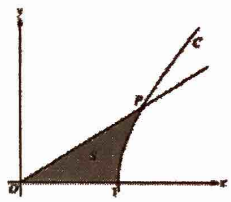
Example 1. 2003 BC Exam Question 3 (calculator allowed)

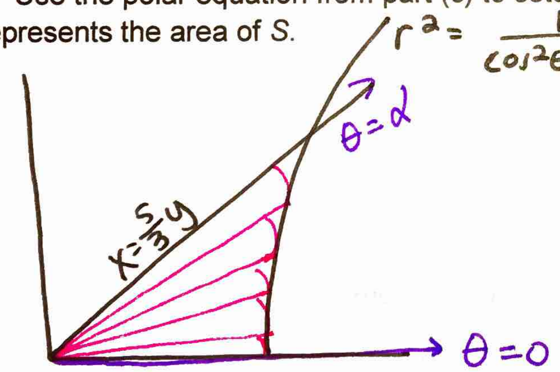
The figure shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.



c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$.

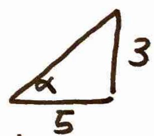
$$x^2 - y^2 = 1$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$
$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$$

d) Use the polar equation from part (c) to setup an integral expression with respect to the polar angle θ that represents the area of S.



Find α:

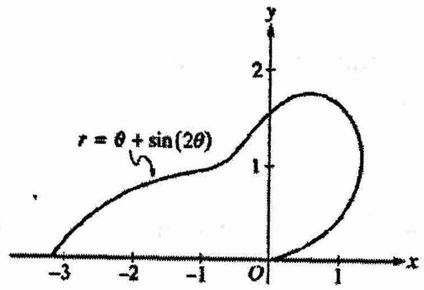
$$x = \frac{5}{3}y$$
$$y = \frac{3}{5}x$$



$$\alpha = \arctan\left(\frac{3}{5}\right)$$

$$\text{Area} = \int_0^{\arctan(3/5)} \frac{1}{2} R^2 \, d\theta = \frac{1}{2}\int_0^{\arctan(3/5)} \frac{d\theta}{\cos^2\theta - \sin^2\theta}$$

Example 2. 2005 BC Exam Question 2 (calculator allowed)



The curve above is drawn in the *xy*-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where *r* is measured in meters and θ is measured in radians. The derivative of *r* with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

a) Find the area bounded by the curve and the *x*-axis

$$\frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta \doteq \boxed{4.382}$$

b) Find the angle θ that corresponds to the point on the curve with *x*-coordinate -2.

$x = -2 = r\cos\theta$, $r = \theta + \sin(2\theta)$

$-2 = (\sin(2\theta) + \theta)\cos\theta$

$$0 = \underbrace{(\sin(2\theta) + \theta)\cos\theta + 2}$$

use calculator to find roots of this on $0 \le \theta \le \pi$

$\boxed{\theta \doteq 2.786}$

c) For $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

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if $\frac{dr}{d\theta} < 0$, and $r > 0$, it means that the radius is decreasing as θ increases.

So, as you move counterclockwise on $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, the curve gets closer to the pole.

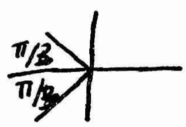
d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

maximum R

$$R = \theta + \sin(2\theta)$$

$$R' = 1 + 2\cos(2\theta) = 0$$

$$\cos(2\theta) = -\frac{1}{2}$$



$$2\theta = \pm\frac{2\pi}{3} + 2\pi k$$

$$\theta = \pm\frac{\pi}{3} + \pi k$$

on $0 \le \theta \le \frac{\pi}{2}$, $\theta = \frac{\pi}{3}$

EXTREME VALUE THM: MAX OCCURS @ CRITICAL VALUES OR AT ENDPTS:

$R(0) = 0 + \sin 0 = 0$

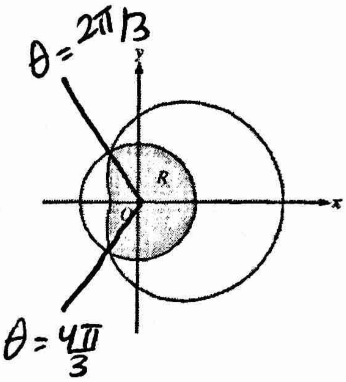
$R(\frac{\pi}{3}) = \frac{\pi}{3} + \sin(\frac{2\pi}{3}) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ ← greatest value.

$R(\frac{\pi}{2}) = \frac{\pi}{2} + \sin(\pi) = \frac{\pi}{2}$

∴ The ~~greatest~~ point w/ greatest distance from origin is @ $\theta = \frac{\pi}{3}$

Example 3. 2007 BC Exam Question 3. (calculator allowed).

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

Circle on $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ + cardioid on $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$

$$= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} 2^2\,d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3+2\cos\theta)^2\,d\theta$$

≈ 10.370

b) A particle moving with nonzero velocity along the polar curve $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

$$\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = \left.0 + 2(-\sin\theta)\right|_{\theta=\pi/3} = 0 - 2\sin\frac{\pi}{3} = 0 - \sqrt{3} = -\sqrt{3}$$

since $\frac{dr}{d\theta} < 0$ and $r > 0$, ~~R~~ is decreasing, meaning that the particle is moving closer to the pole/origin at $\theta = \pi/3$

c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

$y = r\sin\theta$

$y = (3 + 2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (-2\sin^2\theta) + (3 + 2\cos\theta)\cos\theta$

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = \left(-2\sin^2\frac{\pi}{3}\right) + \left(3 + 2\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}\right)$

$= -\frac{3}{2} + (4)\left(\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

If $\frac{dy}{dt} > 0$ and $y > 0$, y is increasing and the particle is moving away from the x-axis.