Review - Polar Coordinate Calculus

Polar coordinates are another way of expressing points in a plane. Instead of being centered at an origin and moving horizontally or vertically, polar coordinates are centered at the pole and measure a radius from the pole at a given angle. The beginning angle of zero radians corresponds to the positive *x*-axis. Although polar functions are differentiated in r and θ , the coordinates and slope of a line tangent to a polar curve are given in rectangular coordinates.

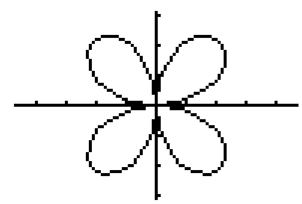
Tangent lines

You should remember the following conversions:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

They are necessary to find the derivative of a polar curve in *x*- and *y*-coordinates. The derivative is done parametrically so that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$. Remember, when you differentiate in *x* or *y*, remember to use the product formula.

Example: Find the slope of the rose curve $r = 3\sin 2\theta$ at the point where $\theta = \frac{\pi}{6}$ and use it to write an equation for the line tangent to the graph.



Solution

First, find the coordinates of the point

$$x = r\cos\theta \Rightarrow x = 3\sin 2\theta\cos\theta\Big|_{\theta=\frac{\pi}{6}} = \frac{9}{4}$$

$$y = r\sin\theta \Rightarrow y = 3\sin 2\theta\sin\theta\Big|_{\theta=\frac{\pi}{6}} = \frac{3\sqrt{3}}{4}$$

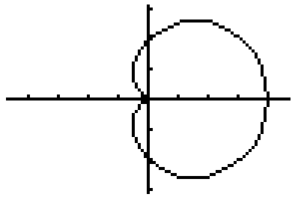
Next, find the derivative and evaluate it at the given angle

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin 2\theta\cos\theta + 6\cos 2\theta\sin\theta}{-3\sin 2\theta\sin\theta + 6\cos 2\theta\cos\theta}$$

$$\frac{3\sin 2\theta\cos\theta + 6\cos 2\theta\sin\theta}{-3\sin 2\theta\sin\theta + 6\cos 2\theta\cos\theta}\Bigg|_{\theta=\frac{\pi}{6}} = \frac{15/4}{9\sqrt{3}/4} = \frac{5\sqrt{3}}{9}$$

The tangent line is $y - \frac{3\sqrt{3}}{4} = \frac{5\sqrt{3}}{9}\left(x - \frac{9}{4}\right)$.

Area enclosed by polar curves



The area enclosed by a polar curve is found by $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \; d\theta$

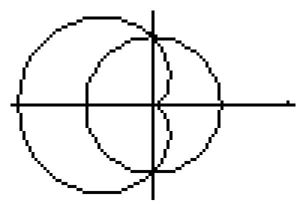
Remember the $d\theta$ refers to a thin pie-shaped sector.

Example: Find the area enclosed by the cardioid $r = 2 + 2\cos\theta \;\; from\, 0 \le \theta \le 2\pi.$

Solution

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(2 + 2\cos\theta\right)^2 \; d\theta \approx 18.8496$$

Area between two polar curves



The area enclosed between two polar curves is given by $A = \frac{1}{2}\int_{\alpha}^{\beta} \left(\left(r_{outer}\right)^2 - \left(r_{inner}\right)^2\right) d\theta$. Notice that the integrand is the difference of the radii squared, not the square of the difference of the radii.

Example: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta.$

Solution

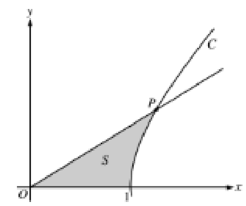
The outer curve is the circle $r = 1$ and the inner curve is the cardioid $r = 1 - \cos\theta.$ The points of intersection are $\theta = -\frac{\pi}{2} \;\; and \;\; \theta = \frac{\pi}{2}.$

$$A = \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\left(1\right)^2 - \left(1 - \cos\theta\right)^2\right) d\theta \approx 1.215.$$

If the region has symmetry, you can find the area of one segment and then multiply it by the number of symmetric segments you have.

Example 1. 2003 BC Exam Question 3 (calculator allowed)

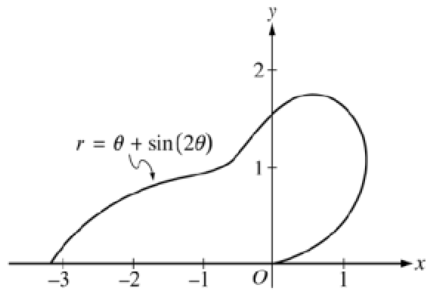
The figure shows the graphs of the line $x = \frac{5}{3}y$ and the curve *C* given by $x = \sqrt{1+y^2}$. Let *S* be the shaded region bounded by the two graphs and the *x*-axis. The line and the curve intersect at point *P*.



c) Curve *C* is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$.

d) Use the polar equation from part (c) to setup an integral expression with respect to the polar angle θ that represents the area of *S*.

Example 2. 2005 BC Exam Question 2 (calculator allowed)



The curve above is drawn in the *xy*-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi,$ where *r* is measured in meters and θ is measured in radians. The derivative of *r* with respect to θ is given by $\dfrac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

a) Find the area bounded by the curve and the *x*-axis

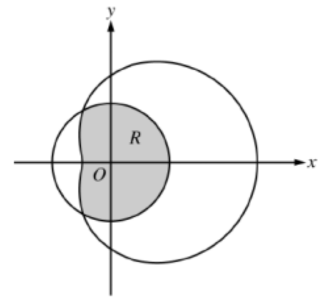
b) Find the angle θ that corresponds to the point on the curve with *x*-coordinate -2.

c) For $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about *r*? What does this fact say about the curve?

d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

Example 3. 2007 BC Exam Question 3. (calculator allowed).

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



a) Let *R* be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of *R*.

b) A particle moving with nonzero velocity along the polar curve $r = 3 + 2\cos\theta$ has position $\left(x(t), y(t)\right)$ at time *t*, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.